Comparison of some strategies for Restarting GMRES

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Introduction

The Restarted Generalized Minimal Residual Method (GMRES(m)) is one of the most successful methods for solving linear systems of equations $Ax = b$, where A is a nonsymmetric sparse matrix[6]. At each cycle, GMRES(m) uses the residual at the previous cycle as starting guess, and constructs a Krylov subspace of dimension m with $m \ll n$ (where n is the dimension of the linear system) for computing a new residual, which is used as the starting residual for the next cycle, i.e., the next call to a GMRES routine. Rate of GMRES(m) convergence depends on an appropriate selection of the restarting parameter m. In this context several algorithms have been proposed for choosing statically and dynamically the parameter m or introducing vectors for enriching the subspace [2, 3].

Models comparison

In this work we compare the performance of the proposed method called Adaptive-GMRES with the standard GMRES(m) and other methods that try to acelerate the convergence. These methods are:

- GMRES-E(m,d) method proposed by R. B. Morgan [5], improves the convergence by appending d approximate eigenvectors to the Krylov subspace.
- LGMRES(m,l) method proposed by A. H. Baker [1], improves the convergence by appending l error approximation vectors.

Control formulation

At each cycle, GMRES(m) finds a solution of the form

$$x_j = x_{j-1} + V_m y_j, \tag{1}$$

where x_{j-1} is the previous approximate solution of x, and the residual is $r_{j-1} = b - Ax_{j-1}$; then V_m is a $n \times m$ matrix where its columns form an orthogonal basis of the Krylov Subspace $\mathcal{K}_m(A, r_{j-1}) \equiv span\{r_{j-1}, Ar_{j-1}, A^2 r_{j-1}, \ldots, A^{m-1} r_{j-1}\}$. Furthermore, y_j minimize the l_2-norm of the residual $\|r_j\|_2 = \|b - A(x_{j-1} + V_m y_j)\|_2 = \|\beta e_1 - \tilde{H}_m y_j\|_2$.

When the l_2-norm of the last y_j is very small, then $x_j \approx x_{j-1}$ and stagnation occurs. Hence, the proposed strategy called Adaptive-GMRES(m) consists in modifying the value of m before each restarted cycle.

An example of a proportional controller for m is given by:

$$m_j = m_{j-1} + u_j, \tag{2}$$

where

$$u_j = \begin{cases} 1 \;\; if \|y_j\|_2 < \epsilon_0 \\ 0 \;\; if \|y_j\|_2 \geq \epsilon_0 \end{cases} \tag{3}$$

Selected problems

Partial tests on classic problems from the SuiteSparse matrix collection [4] are performed. For the Group A, GMRES(m) converges before 2000 restart cycles, and for Group B, GMRES(m) does not converge before 2000 restart cycles. n is the size of A, nnz is the number of nonzero elements in A and $cond(A)$ is the condition number of A.

	Problem Group A	n	nnz	Application area	cond(A)
A1	add20	2395	17319	circuit simulation problem	12047,1
A2	cavity05	1182	32632	computational fluid dynamics problem	577065
A3	circuit_2	4510	21199	circuit simulation problem	131925
A4	fpga_trans_01	1220	7382	circuit simulation problem	12214,3
A5	memplus	17758	99147	circuit simulation problem	129436
A6	sherman4	3312	20793	computational fluid dynamics problem	2178.63

	Problem Group B	n	nnz	Application area	cond(A)
B1	sherman3	5005	20033	computational fluid dynamics problem	5,01425e+17
B2	sherman5	3312	20793	computational fluid dynamics problem	1,87941e+05
B3	TSOPF_RS_b162_c1	5374	205399	power network problem	8,59445e+07
B4	young3c	841	3988	acoustics problem	9298,3

Algorithm settings

For comparison purposes $Ax = b$ was solved using: GMRES(m), GMRES-E(m,d), LGMRES(m,l) and the Adaptive-GMRES(m). Algorithms settings: initial solution is $x_0 = 0$, stopping criterion is $\frac{\|r_j\|_2}{\|r_0\|_2} < 10^{-6}$ or a maximum of 2000 restart cycles. GMRES(m): $m = 30$. LGMRES(m,l): $m = 27$, $l = 3$. GMRES-E(m,d): $m = 27$, $d = 3$. Adaptive-GMRES(m) has initial restart parameter $m_0 = 30$ and $\epsilon_0 = 10^{-10}$. The reported times are the average of 5 runs.

Numerical results

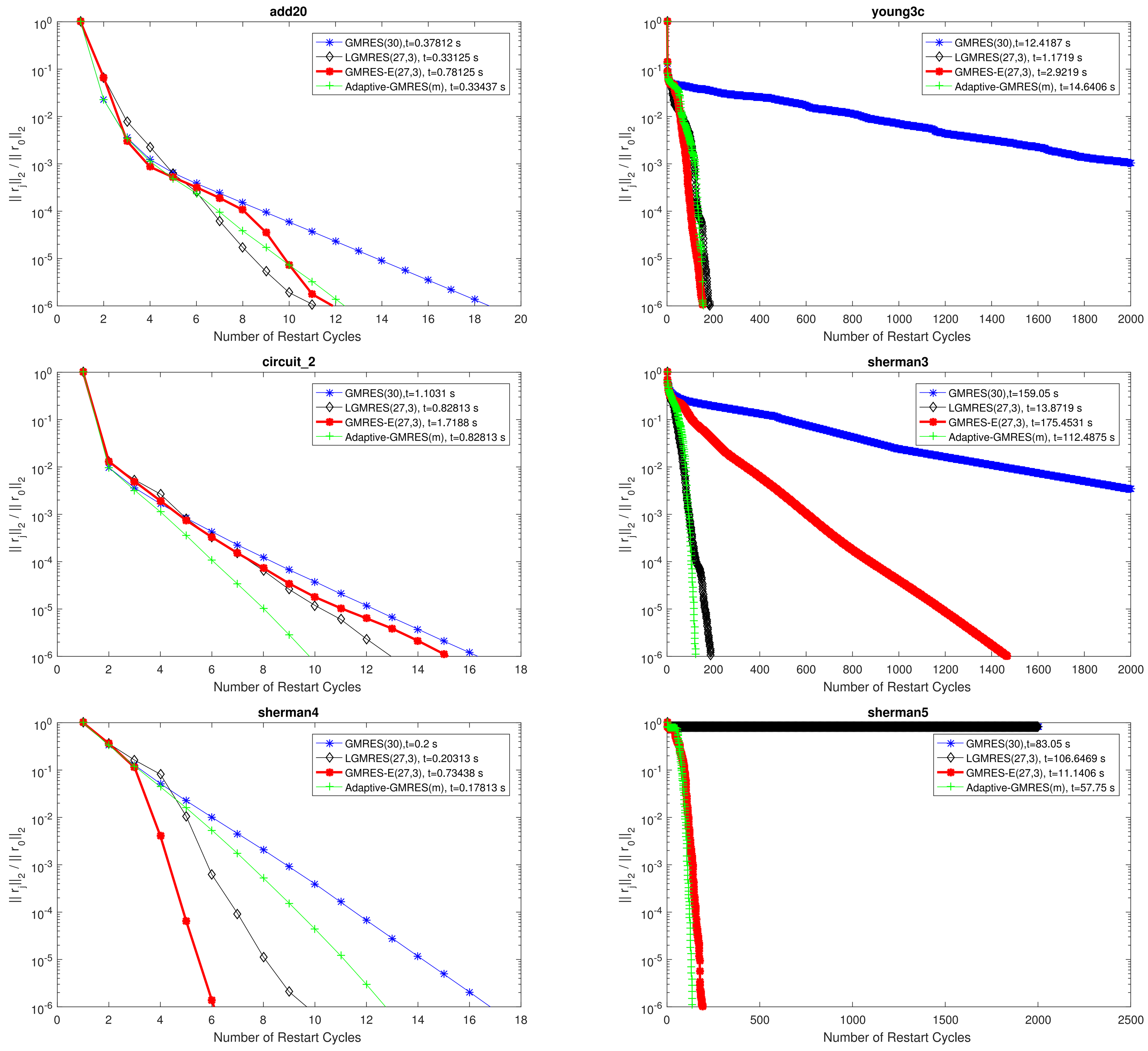

Figure 1: Examples of solved problems. **(Left column:)** Problem Group A, GMRES(m) converges before 2000 restart cycles. **(Right column:)** Problem Group B, GMRES(m) does not converge before 2000 restart cycles.

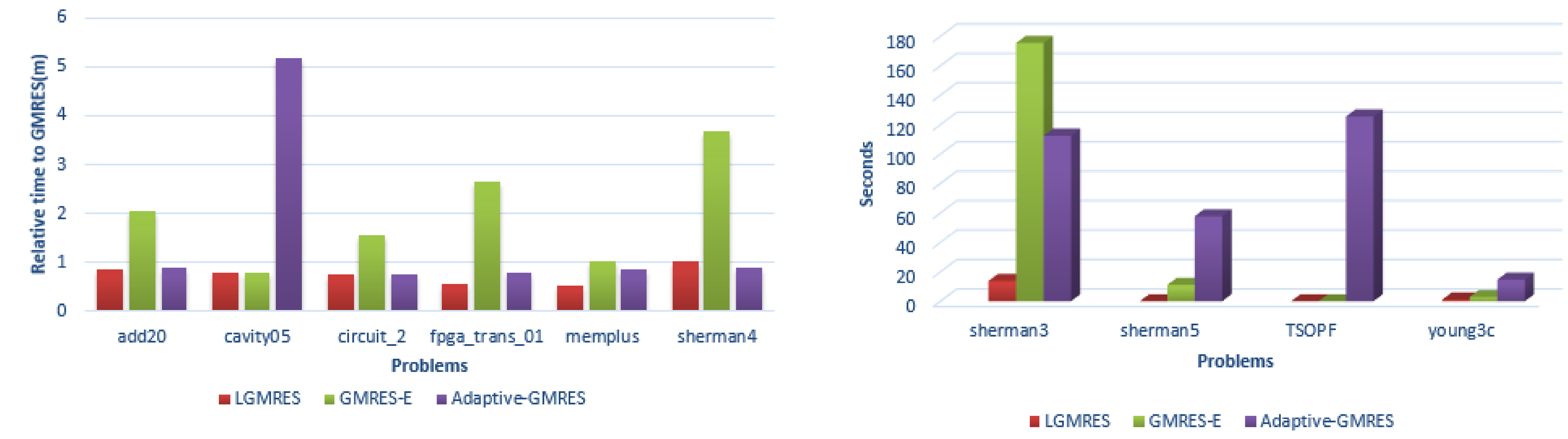

Figure 2: **(Left:)** Execution time ratio of the selected algorithms -relative to GMRES(m)- for Problem Group A. **(Right:)** Execution time ratio of the selected algorithms for Problem Group B.

Conclusion

The Adaptive-GMRES(m) method has good convergence properties for both groups of problems. We show that increasing the value of m when we have stalling improves the information in the restarted GMRES. The criterion of increasing the value of m when the value of $\|y_j\|_2$ is small, allows to avoid slow convergences and stagnations in standard GMRES(m). Future work may find better heuristics for u_j in order to reduce execution times.

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